Combinatorial Matrices from Certain Balanced Incomplete Block Designs

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Abstract

Some combinatorial matrices are obtained from affine resolvable and near resolvable designs. These matrices are used in the construction of group divisible designs and Latin square type designs which are important families of two associate classes partially balanced incomplete block designs.

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1. Introduction

It is well known that combinatorial matrices and design theory are closely related. Dey (1977) used matrix approaches in the construction of group divisible designs for the first time. Later Gibbons and Mathon (1987), Sarvate and Seberry (1998), Kharaghani and Suda (2018), Saurabh *et al.* (2021) and Saurabh and Sinha (2021, 2022, 2023) used generalised Bhaskar Rao designs, difference matrices, generalized Hadamard matrices for the constructions of resolvable group divisible designs and Latin square type designs. Group divisible and Latin square type designs are useful in the construction of low – density parity – check (LDPC) codes [see Xu *et al.* (2015)], in group testing experiments, sampling and intercropping [see Raghavarao and Padgett (2005)]. Raghavarao and Singh (1975) discussed the use of Latin square type designs in cluster sampling. Applications of weighing matrices in optimal weighing designs and optical multiplexing may be found in Koukouvinos and Seberry (1997).

Combinatorial matrices have been studied by Launey (1989, 2007), Jungnickel (1979), Ionin and Kharghani (2007), Abel *et al.* (2004) and Tonchev (2009), among others. A survey on generalised Bhaskar Rao designs over groups of orders ≤ 100 may be found in Abel *et al.* (2010).

Shrikhande (1954) constructed group divisible designs from affine resolvable balanced incomplete block designs (BIBDs). Later Sinha (1988) constructed rectangular Hadamard matrices from resolvable BIBDs and Hurd and Sarvate (2000) obtained *c*–Bhaskar Rao designs from balanced incomplete block designs. Here difference matrices and generalised weighing matrices have been obtained from affine resolvable BIBDs and near resolvable designs respectively. K numbers used in this paper are from Kageyama (1972).

2. Preliminaries

2. 1 Balanced incomplete block design

A *balanced incomplete block design* (BIBD) or a $2 - (v, k, \lambda)$ design is an arrangement of v elements in b blocks each of size $k(< v)$ such that
a. Every element is replicated r times and
b. Any pair of distinct elements occur together in λ blocks.

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The integers v, b, r, k, λ are called parameters of the BIBD and they satisfy the relations: $bk = vr; r(k - 1) = \lambda(v - 1), v \leq b$ (Fisher's inequality). A BIBD is symmetric if $v = b$ and is $self-complementary$ if $v = 2k$.

A BIBD is *resolvable* if the blocks can be partitioned into $'r'$ classes, called *resolution classes* such that each element is replicated exactly once in each resolution class. Further if any two blocks from two distinct resolution classes intersect in same number of elements, say q then the BIBD is said to be *affine resolvable* with parameters v, b, r, k, λ, q.

2. 2 Near resolvable designs

A BIBD D with parameters v, b, r, k, λ is said to be *near resolvable, NRB $(v, k, k - 1)$* if its blocks can be partitioned into v classes such that for each element θ of D there is precisely one class which does not contain θ in any of its blocks and each class contains $v - 1$ distinct elements. Such classes are called partial resolution classes or near parallel classes.

2. 3 Generalised weighing matrix

Let G be a multiplicative group of order g. A *generalised Weighing matrix* is a $v \times b$ matrix M with entries from $G \cup \{0\}$ where such that the inner product of any pair of distinct rows of M contains every element of G same number of times, say μ.

2. 4 Generalised Bhaskar Rao designs and Difference Matrices

Let G be a multiplicative group of order g. A *generalised Bhaskar Rao design GBRD* $(v, b, r, k, \lambda; G)$ over G is a $v \times b$ array with entries from $G \cup \{0\}$ such that:

a. Each row contains exactly r group elements;
b. Each column contains exactly k group elements;
c. In the inner product $\{x_i y_i^{-1} : i = 1, 2, \ldots, b; x_i, y_i \neq 0\}$ of each pair of distinct rows $(x_1, x_2, \ldots, x_b)$ and $(y_1, y_2, \ldots, y_b)$, each group element occurs exactly λ/g times.

Further if we replace the nonzero entries of a *GBRD* $(v, b, r, k, \lambda; G)$ by unity then we obtain the incidence matrix of a BIBD with parameters: v, b, r, k, λ. A *difference matrix* $D(k, \lambda g; G)$, is a GBRD$(k, \lambda g, \lambda g, k, \lambda g; G)$ i.e. difference matrices are GBRD's with non–zero entries. If the difference matrix is square then it is known as a *generalised Hadamard matrices* over G of order λg and index λ, GH $(\lambda g; G)$ i. e. a generalised Hadamard matrix is a difference matrix with maximum number of rows.

3. Construction Theorems

Theorem 1: The existence of an affine resolvable BIBD with parameters $v = qs^2, b, r, k, \lambda, q$ implies the existence of a difference matrix, $D(r, qs^2; G)$ where G is a cyclic group of order $s = b/r$.

Proof: Let $G = \{1, \alpha, \alpha^2, \ldots, \alpha^{s-1}\} = \langle \alpha \rangle$ be a cyclic group of order s under multiplication and $R_1, R_2, R_3, \ldots, R_r$ be the resolution classes of an affine resolvable BIBD D with parameters $v = qs^2, b, r, k, \lambda, q$. Further let $B_i^1, B_i^2, B_i^3, \ldots, B_i^s$ be the randomly chosen blocks from i^{th} resolution class $R_i (1 \leq i \leq r)$ of D. We construct $r \times qs^2$ matrix $D(r, qs^2; G)$ as follows:

(a) The elements of B_i^1 are replaced by 1, the elements of B_i^2 are replaced by $\alpha,\dots$, the elements of B_i^s are replaced by α^{s-1} which form the ith row of $D(r, qs^2; G)$. We repeat this process for each resolution class of D.

(b) The columns of $D(r, qs^2; G)$ are indexed as $1,2,3,\dots,qs^2$ and an element $\alpha^{j-1}(1 \le j \le s)$ of G is placed just below j if $j \in B_i^j$.

Since any two blocks from two different resolution classes intersect in q elements, inner product of any two distinct rows of $D(r, qs^2; G)$ contain each element of G, qs times. Hence $D(r, qs^2; G)$ is the desired difference matrix.

Example 1: Consider an affine resolvable BIBD K17: $v = 16, b = 20, r = 5, k = 4, \lambda = 1, q = 1$ whose resolution classes are:

RI: [(1 2 3 4) (5 6 7 8) (9 10 11 12) (13 14 15 16)];

RII: [(1 5 9 13) (2 6 10 14) (3 7 11 15) (4 8 12 16)];

RIII: [(1 6 11 16) (2 5 12 15) (3 8 9 14) (4 7 10 13)];

RIV: [(1 7 12 14) (2 8 11 13) (3 5 10 16) (4 6 9 15)];

RV: [(1 8 10 15) (2 7 9 16) (3 6 12 13) (4 5 11 14)]

Then a difference matrix $D(5,16; G)$ over a cyclic group $G = \{1, \alpha, \alpha^2, \alpha^3\}$ is:

$$D(5,16; G) = \begin{bmatrix} 1 & 1 & 1 & 1 & \alpha & \alpha & \alpha & \alpha & \alpha^2 & \alpha^2 & \alpha^2 & \alpha^2 & \alpha^3 & \alpha^3 & \alpha^3 & \alpha^3 \\ 1 & \alpha & \alpha^2 & \alpha^3 & 1 & \alpha & \alpha^2 & \alpha^3 & 1 & \alpha & \alpha^2 & \alpha^3 & 1 & \alpha & \alpha^2 & \alpha^3 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha & 1 & \alpha^3 & \alpha^2 & \alpha^2 & \alpha^3 & 1 & \alpha & \alpha^3 & \alpha^2 & \alpha & 1 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^2 & \alpha^3 & 1 & \alpha & \alpha^3 & \alpha^2 & \alpha & 1 & \alpha & 1 & \alpha^3 & \alpha^2 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^3 & \alpha^2 & \alpha & 1 & \alpha & 1 & \alpha^3 & \alpha^2 & \alpha^2 & \alpha^3 & 1 & \alpha \end{bmatrix}.$$

As a particular case of Theorem 1, we have

Corollary 1: The existence of an affine resolvable BIBD with parameters $v = 2k, b = 2r, r, k = 2q, \lambda, q$ implies the existence of a rectangular Hadamard matrix of order $r \times 4q$.

Example 2: Consider an affine resolvable BIBD K5: $v = 8, b = 14, r = 7, k = 4, \lambda = 3, q = 2$ whose resolution classes are:

[(1 2 4 7) (3 5 6 8)]; [(1 2 3 5) (4 6 7 8)]; [(1 5 7 8) (2 3 4 6)]; [(1 2 6 8) (3 4 5 7)];

[(1 4 5 6) (2 3 7 8)]; [(1 3 4 8) (2 5 6 7)]; [(1 3 6 7) (2 4 5 8)]

Then a rectangular Hadamard matrix of order 7×8 is:

$$\begin{bmatrix} 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

Theorem 2: The existence of a NRB$(v, k, k - 1)$ implies the existence of a $v \times (v + 1)$ generalised weighing matrix with entries from a cyclic group G of order s where $s = \frac{v-1}{k}$.

Proof: Let $G = \{1, \alpha, \alpha^2, \ldots, \alpha^{s-1}\} = \langle \alpha \rangle$ be a cyclic group of order s under multiplication and $R_1, R_2, R_3, \ldots, R_v$ be the partial resolution classes of a NRB$(v, k, k-1)$. Further let $B_i^1, B_i^2, B_i^3, \ldots, B_i^s$ be the randomly chosen blocks from i^{th} resolution class: $R_i(1 \leq i \leq v)$. Then using property of near resolvability, we have

$$\sum_{\substack{1 \leq i \neq j \leq v \\ 1 \leq k \neq l \leq s}} \left| B_i^k \cap B_j^l \right| = v - 2 \tag{1}$$

We construct a matrix M of order $v \times v$ whose i^{th} row is obtained as follows:

(a) The missing element is represented by 0;
(b) The elements of the block B_i^k of $R_i(1 \leq i \leq v)$ is replaced by $\alpha^{k-1}(1 \leq k \leq s)$.

Then a matrix N is obtained by adjoining a column of ones to M. Then using (1), it may be easily verified that the inner product of any two distinct rows of N contains any element of G; $\frac{v-2+1}{s} = \frac{v-1}{s} = k$ times. Hence N is the desired $v \times (v+1)$ generalised weighing matrix.

Remark: For $v = 2k+1$, Theorem 2 yields a rectangular weighing matrix. A table of NRB$(v, k, k-1)$ under the range $15 \leq v \leq 200; 3 \leq k \leq 97$ may be found in Furino *et al.* (1996). Although from a generalized weighing matrix of order $v \times b$, a $v \times (v+1)$ matrix for all $v+1 \leq b$ can be constructed, the above method of construction may yield non $-$ isomorphic solutions.

Example 3: Consider a NRB (7,2,1) whose partial resolution classes are given as:

Missing element	{1}	{2}	{3}	{4}	{5}	{6}	{7}
	(2, 7)	(1, 3)	(1, 5)	(1, 7)	(1, 2)	(1, 4)	(1, 6)
Blocks	(3, 6)	(4, 7)	(2, 4)	(2, 6)	(3,7)	(2, 3)	(2, 5)
	(4, 5)	(5, 6)	(6, 7)	(3, 5)	(4, 6)	(5, 7)	(3, 4)

Then a 7×8 generalised Weighing matrix $N_{7\times8}$ over a cyclic group $G = \{1, \alpha, \alpha^2\}$ is:

$$N_{7\times8} = \begin{bmatrix} 1 & 0 & 1 & \alpha & \alpha^2 & \alpha^2 & \alpha & 1 \\ 1 & 1 & 0 & 1 & \alpha & \alpha^2 & \alpha^2 & \alpha \\ 1 & 1 & \alpha & 0 & \alpha & 1 & \alpha^2 & \alpha^2 \\ 1 & 1 & \alpha & \alpha^2 & 0 & \alpha^2 & \alpha & 1 \\ 1 & 1 & 1 & \alpha & \alpha^2 & 0 & \alpha^2 & \alpha \\ 1 & 1 & \alpha & \alpha & 1 & \alpha^2 & 0 & \alpha^2 \\ 1 & 1 & \alpha & \alpha^2 & \alpha^2 & \alpha & 1 & 0 \end{bmatrix}.$$

Example 4: Consider the following solution of NRB (7,3,2) as given in Abel *et al.* (2007, *p.* 124):

Missing element	{0}	{1}	{2}	{3}	{4}	{5}	{6}
Blocks	(1, 2, 4)	(2, 3, 5)	(3, 4, 6)	(0, 4, 5)	(1, 5, 6)	(0, 2, 6)	(0, 1, 3)
	(3, 5, 6)	(0, 4, 6)	(0, 1, 5)	(1, 2, 6)	(0, 2, 3)	(1, 3, 4)	(2, 4, 5)

Then a 7×8 rectangular Weighing matrix $N_{7\times8}$ is:

$$N_{7 \times 8} = \begin{bmatrix} 1 & 0 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 0 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & 0 \end{bmatrix}.$$

4. Conclusion

In this paper difference matrices and generalised weighing matrices have been obtained from affine resolvable BIBDs and near resolvable designs respectively. These matrices are of combinatorial as well as of statistical interest. It would be interesting to obtain cocyclic generalised Hadamard matrices, balanced generalised weighing matrices and other combinatorial matrices from certain block designs. A generalised Hadamard matrix gives a complete resolvable transversal design whereas a cocyclic generalised Hadamard matrix is equivalent to semi–regular central relative difference set. For details on these matrices, see Launey (2007) and Ionin and Kharghani (2007).

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